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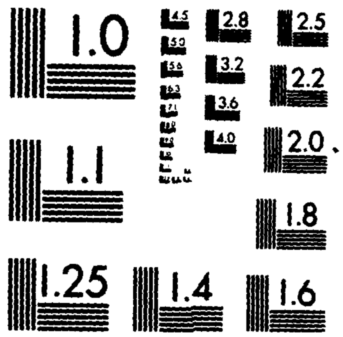

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Large Deviation Local Limit Theorems For Random Vectors

Narasinga Rao Chaganty and Jayaram Sethuraman †

By

Department of Mathematics
Old Dominion University
Norfolk, Virginia 23508.

Department of Statistics
Florida State University
Tallahassee, Florida 32306.

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Abstract

In this paper we present a survey of large deviation local limit theorems for random vectors. We then establish a more extensive large deviation local limit theorem that requires somewhat weaker conditions even in the special cases proved earlier.

1. Introduction

Suppose that $\{T_n\}$ is a sequence of random variables in R_m, the m- dimensional Euclidean space such that $T_n/\sqrt{n}$ converges in distribution to a non-singular multivariate normal distribution. A local limit theorem is a result concerning the limit of the probability density function (p.d.f.) of $T_n/\sqrt{n}$ at a fixed point $x \in R_m$. A large deviation local limit theorem is a similar result concerning the p.d.f. of $T_n/\sqrt{n}$ at a point x_n where $x_n \to \infty$ and $x_n = O(\sqrt{n})$. It is more convenient to state that it as a result concerning the p.d.f. of T_n/n at a point x_n where $x_n = O(1)$.

When T_n is the sum of n independent and identically distributed nonlattice valued random vectors $X_1, \ldots, X_n$ with common distribution function (d.f.) F, Richter(1958) established the earliest result on large deviation local limit theorems for T_n/n at a point x_n where $x_n = o(1)$ and $\sqrt{n}x_n > 1$. This is stated as Theorem 2.5 in Section 2. The main conditions imposed on the existence of a moment generating function ϕ (m.g.f.) for X_1 and some integrability properties of ϕ, which in turn imply the existence of a p.d.f. for T_n/n.

Phillips(1977) extended the above theorem for arbitrary nonlattice random vectors T_n. The conditions imposed here were on the m.g.f. of T_n. This result is stated in Theorem 3.1 in Section 3. The asymptotic expression for the p.d.f. of T_n/n at x_n given by both the above authors involved the so called Cramér series. As we had pointed out for the one-dimensional case in Chaganty and Sethuraman(1985), this can be greatly simplified by the use of the large deviation rate γ_n of T_n/n, which exists under the already imposed assumptions.

Furthermore both the authors impose the condition $x_n = o(1)$. In Section 3, we dispense with this condition and require only that $x_n = O(1)$ which corresponds to an arbitrary large deviation. Borokov and Rogozin(1965) have established a large deviation local limit theorem for sums of i.i.d. nonlattice randon vectors requiring only that $x_n = O(1)$. This is stated as Theorem 2.6. Our main result stated in Theorem 3.2 is in the same spirit and holds for arbitrary nonlattice random vectors and requires conditions on the m.g.f. of T_n which are weaker than those imposed in Theorems 2.6 and 3.1.

The organization of this paper is as follows. Section 2 consists of some preliminaries and lists the results of Richter(1958) and Borokov and Rogozin(1965). Section 3 contains the result of Phillips(1977) and our main result. Section 4 gives the analogous results for lattice valued random vectors.

2. Large deviation local limit theorems for i.i.d. nonlattice random vectors

In this section we will describe several well known local limit theorems for random vectors in R_m, the m-dimensional Euclidean space with real components. In order to standardize the notation used by several authors, we begin with a description of the notations used throughout this paper. The space of m-dimensional vectors with complex components will be denoted by C_m. We write just R for R_1 and C for C_1. Throughout this paper we denote points of R_m by t, u, r, s etc., and a point of R_m with nonnegative integer components by α and a point of C_m is denoted by z. The j^{th} component of a vector x is denoted by x_j. We shall further use the following standard notation:

$$
\begin{aligned}
z^\alpha &= z_1^{\alpha_1} z_2^{\alpha_2} \dots z_m^{\alpha_m}, \\
dt &= dt_1 dt_2 \dots dt_m, \\
< z, t > &= z_1 t_1 + z_2 t_2 + \cdots + z_m t_m, \\
\overline{z} &= (\overline{z}_1, \overline{z}_2, \dots, \overline{z}_m), \\
az &= (az_1, az_2, \dots, az_m) \qquad \text{when} \quad a \in R, \\
\text{and} \qquad \underline{1} &= (1, \dots, 1).
\end{aligned} \tag{2-1}
$$

When f is a complex valued function defined on R_m and j, k are positive integers we write

$$
\begin{aligned}
D_j f(t) &= \frac{df(t)}{dt_j}, \\
D_{jk} f(t) &= \frac{d^2 f(t)}{dt_j dt_k}, \\
\nabla f(t) &= (D_1 f(t), \dots, D_m f(t)), \\
\text{and} \qquad \nabla^2 f(t) &= \big((D_{jk} f(t))\big).
\end{aligned} \tag{2-2}
$$

The determinant of the matrix $\nabla^2 f(t)$ is denoted by $\mid \nabla^2 f(t) \mid$.

Definition 2.1. A polydisc $s(z, r)$ of radius $r = (r_1, ..., r_m)$ around a point $z = (z_1, \dots, z_m)$ is defined as

$$
s(z, r) = s(z_1, r_1) \times s(z_2, r_2) \times \cdots \times s(z_m, r_m) \tag{2-3}
$$

where

$$s(z_j, r_j) = [z_j' : \mid z_j' - z_j \mid < r_j], \qquad j = 1, \ldots, m.$$

The closure of $s(z,r)$ is denoted by $\bar{s}(z,r)$.

Definition 2.2. A complex valued function f is said to be holomorphic at a point z_0 of C_m if there is an $r > 0$ such that

$$f(z) = \sum_{|\alpha| \geq 0} a_\alpha (z - z_0)^\alpha, \qquad \text{for } z \in s(z_0, r) \tag{2-4}$$

where $a_\alpha \in C_m$; and the above power series is absolutely convergent.

Let $(z + re^{i\theta}) = (z_1 + r_1 e^{i\theta_1}, \ldots, z_m + r_m e^{i\theta_m})$ for $r = (r_1, \ldots, r_m)$ and $\theta = (\theta_1, \ldots, \theta_m)$. The following theorem can be found in Vladimirov(1966) pp 30-31.

Theorem 2.3.(Cauchy). Suppose that a function $f(z)$ is holomorphic and that it is bounded in the closed polydisc $\bar{s}(z_0, r)$. Then the coefficient a_α in the expansion of $f(z)$ is given by

$$a_\alpha = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{r^\alpha} \exp(-i < \theta, \alpha >)\, d\theta. \tag{2-5}$$

Consequently,

$$|a_\alpha| \leq \frac{1}{r^\alpha} \sup_{z \in \bar{s}(z_0, r)} |f(z)|. \tag{2-6}$$

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed m-dimensional nonlattice random vectors with common distribution function F. Let $E(X_1) = 0$ and $Cov(X_1) = V$ be the covariance matrix of X_1. Let the moment generating function of X_1 be given by

$$\phi(z) = \int_{R_m} \exp(< z, y >)\, dF(y), \qquad \text{for} \quad z \in C_m. \tag{2-7}$$

When $z = s$ is real then $\phi(z)$ as a function of s is the usual moment generating function and when $z = it$ is purely imaginary, $\phi(z)$ as a function of t is the characteristic function of X_1. Let $\overline{X}_n = S_n/n$, where $S_n = X_1 + \ldots + X_n$ is the n^{th} partial sum. We assume that the following conditions are satisfied for the distribution of X_1:

I. There exists a *nonempty open set* $A \subset R_m$ such that $\phi(t)$ is finite for $t \in A$.

II. There exists a natural number n_0 such that $\overline{X}_{n_0}$ has a bounded density p_{n_0}.

The following theorem can be found in Chapter 4 of Bhattacharya and Ranga Rao (1976). It provides necessary and sufficient conditions for condition II to hold:

Theorem 2.4. Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d. nonlattice random vectors with values in R_m. Assume that $E(X_1) = 0$, $Cov(X_1) = V$, where V is a symmetric positive definite matrix. Let $\phi(z)$ be the moment generating function of X_1. Then Condition II is equivalent to each of the statements stated below:

(i) There exists $p \geq 1$ such that

$$\int_{R_m} |\phi(it)|^p \, dt < \infty \tag{2-8}$$

(ii) There exists n_0 such that for $n \geq n_0$, $\overline{X}_n$ has a density p_n and

$$\lim_{n \to \infty} \sup_{x \in R_m} \left|\frac{1}{\sqrt{n}} p_n\left(\frac{x}{\sqrt{n}}\right) - n(x)\right| = 0, \tag{2-9}$$

where $n(x)$ is the multivariate normal density with mean zero and covariance matrix V.

Let $\psi(s) = \log \phi(s)$, $s \in A$, be the cumulant generating function of X_1. Let the large deviation rate function be given by

$$\gamma(u) = \sup_{s \in \overline{A}} [< u, s > -\psi(s)], \qquad u \in R_m. \tag{2-10}$$

Since $\overline{A}$ is a closed subset of R_m, the supremum is always attained, i.e., for any $u \in R_m$ we can find $\tau(u) \in \overline{A}$ such that

$$u = \nabla\psi(\tau(u)) \qquad \text{and} \tag{2-11}$$

$$\gamma(u) = [< u, \tau(u) > -\psi(\tau(u))], \tag{2-12}$$

where $\nabla\psi = (D_1\psi, \ldots, D_m\psi)$ is the vector of first order partial derivatives. Let us denote by $B \subset R_m$, the set of values of u for which $\tau(u) \in A$. Let A_1 be an arbitrary closed bounded subset of A and $B_1 \subset B$ be the image of A_1 induced by the mapping $\tau^{-1}(\,.)$.

Richter(1958) obtained the fundamental large deviation local limit theorem for the sample mean of i.i.d. nonlattice random vectors which satisfy conditions I and II. This extended his earlier work for real valued random variables (see Richter(1957)). We state Richter's result in the next theorem:

Theorem 2.5.(Richter). Let $X_1, X_2, \ldots$ be i.i.d. nonlattice random vectors with values in R_m. Assume that $E(X_1) = 0$ and $Cov(X_1) = V$ is a positive definite matrix. Let $x_n = (x_{1n}, \ldots, x_{mn})$ be such that $\sqrt{n}x_{jn} > 1$ and $x_{jn} \to 0$ as n $\to \infty$ for all $1 \leq j \leq m$. If X_1 satisfies conditions I and II then the density $p_n(x)$ of $\overline{X}_n$ exists and

$$p_n(x_n) = \frac{n^{m/2}}{(2\pi)^{m/2}|V|^{1/2}} \exp(-n\gamma(x_n))\left[1 + O(\| x_n \|)\right] \tag{2-13}$$

where $\gamma(\ .)$ is the large deviation rate function of X_1.

The conclusion (2-13) in Theorem 2.5 is the same as (1) in Richter's(1958) paper. We have re-written the infinite series appearing in the asymptotic expression for the density in Richter's paper in terms of the large deviation rate function and displayed its crucial role. This simplification has not been noticed by authors in the area of large deviations before Chaganty and Sethuraman(1985) for the case of real valued random variables. Borokov and Rogozin(1965) generalized Theorem 2.5 for sequences $\{y_n\}$ which may not converge to zero and obtained sharper estimates for the remainder term and their result is stated below.

Theorem 2.6.(Borokov and Rogozin). Let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d. random vectors satisfying conditions I and II. Let $\{y_n\}$ be a sequence such that $y_n \in B_1$ for all $n \geq 1$. Let $\rho_n = \tau(y_n) \in A_1$ for $n \geq 1$. Then for any integer $k \geq 1$, the density of the sample mean $\overline{X}_n$ is given by

$$p_n(y_n) = \frac{n^{m/2}}{(2\pi)^{m/2}\ |\nabla^2\psi(\rho_n)|^{1/2}} \exp(-n\gamma(y_n))\left[1 + \sum_{j=1}^{k-1} c_{jn}n^{-j} + O(n^{-k})\right]. \tag{2-14}$$

The coefficients c_{jn} depend only on y_n and on the first $(2j + 2)$ moments of the distribution of X_1 and are uniformly bounded for $y_n \in B_1$.

As an application of Theorem 2.6, Borokov and Rogozin(1965) also obtained integral theorems, that is, estimates of $Pr(\overline{X}_n \in D_n)$, for suitable D_n, as a function of the large deviation rate.

3. Large deviation local limit theorems for nonlattice random vectors.

The results of the previous section are applicable only to sample means of independent and identically distributed random vectors. In applications that arise in mathematical statistics and probability we come across functions which are more general than sample means. In this section we present some generalizations of the theorems of Section 2 which apply to arbitrary sequences of nonlattice random vectors.

Let $T_1, T_2, \ldots$ be a sequence of nonlattice random vectors in R_m. Let $\phi_n(z) = E\exp(< z, T_n >)$ be the moment generating function of T_n, for $z \in C_m$. Assume that $\phi_n(z)$ is holomorphic and non-vanishing in Ω^m, where $\Omega = \{x + iy : x \in I = (-a, a), y \in R\}$ for some $a > 0$. Let $\psi_n(s) = \frac{1}{n}\log\phi_n(s)$, for $\| s \| \leq a$. Let $\nabla\psi_n(s) = (D_1\psi_n(s), \ldots, D_m\psi_n(s))$ be the vector of the first order partial derivatives and $\nabla^2\psi_n(s)$ denote the matrix of second order partial derivatives. The determinant of the matrix $\nabla^2\psi_n(s)$ is denoted by $|\nabla^2\psi_n(s)|$. For $u \in R_m$, we denote the large deviation rate function of T_n/n by

$$\gamma_n(u) = \sup_{s \in I^m} [< u, s > -\psi_n(s)].$$

Theorem 3.1 stated below, is due to Phillips(1977). It extends the results of the previous section for arbitrary sequences T_n, $n \geq 1$, of random vectors.

Theorem 3.1. Let $\{T_n, n \geq 1\}$ be a sequence of nonlattice random vectors such that $E(T_n) = 0$, $Cov(T_n) = n\,\Sigma_n$. Assume that T_n satisfies the following conditions:

(a) The covariance matrix Σ_n has a positive definite limit as $n \to \infty$

(b) There exists positive numbers k_n and K_n such that for $\| z \| < a$ we have

$$k_n \leq |\phi_n(z)| \leq K_n \tag{3-1}$$

(c) There exists $l > 0$ and $b > 0$ such that

$$\int_{\|t\| > Kn^{-l}} |\phi_n(it)|\, dt = O(e^{-bn}) \qquad \text{for all} \quad K > 0. \tag{3-2}$$

Let $x_n = (x_{1n}, \ldots, x_{mn}) \in R_m$ be such that $\sqrt{n}x_{jn} > 1$ and $x_{jn} \to 0$ as $n \to \infty$ for $1 \leq j \leq m$. Then we have

$$f_n(x_n) = \frac{n^{m/2}}{(2\pi)^{m/2}|\Sigma_n|^{1/2}} \exp(-n\gamma_n(x_n))[1 + O(\| x_n \|)] \tag{3-3}$$

where $|\Sigma_n|$ is the determinant of the covariance matrix of $T_n/\sqrt{n}$ and $\gamma_n(.)$ is the large deviation rate function of T_n/n.

We have restated the asymptotic expression for $f_n(x_n)$ in (3-3) in terms of the large deviation rate $\gamma_n(.)$. We notice that Theorem 3.1 requires that $x_n \to 0$, much like Theorem 2.5. We now present our local limit theorem, Theorem 3.2, that does not impose this restriction, which is in the same spirit as Theorem 2.6. After stating and proving Theorem 3.2 we will compare its conditions with those of Theorem 3.1.

Let $\{y_n\}$ be a sequence of vectors in R_m. Theorem 3.2 below is a direct extension of Theorem 2.1 of Chaganty and Sethuraman(1985) to the multi-dimensional case. It provides an asymptotic expression in terms of the large deviation rate function $\gamma_n(.)$, for the density function of T_n/n at the point y_n. Instead of imposing the condition $y_n \to 0$, we will require the condition $\gamma_n(y_n) =< \tau_n, y_n > -\psi_n(\tau_n)$ for some τ_n in $(-a_1, a_1)$ where $a_1 < a$. Let $I_1 = (-a_1, a_1)$ and define

$$G_n(t) = \psi_n(\tau_n) + i < t, \nabla\psi_n(\tau_n) > -\psi_n(\tau_n + it)$$

for t in R_m. Recall that $|t| = |t_1| + |t_2| + \cdots + |t_m|$, for $t = (t_1, \ldots, t_m)$.

Theorem 3.2. Assume that T_n satisfies the following conditions:

(A) There exists $\beta_1 > 0$ such that $|\psi_n(z)| < \beta_1$, for all z in Ω^m and $n \geq 1$.

(B) There exists τ_n in I_1^m such that $\nabla\psi_n(\tau_n) = y_n$ and $\nabla^2\psi_n(\tau)$ is positive definite with the eigenvalues bounded below by $\beta_2 > 0$, for all $\tau \in I^m$ and $n \geq 1$.

(C) There exists $\eta > 0$ such that for any δ, $0 < \delta < \eta$,

$$\inf_{|t| \geq \delta} Real(G_n(t)) = \min[Real(G_n(\delta\underline{1}), Real(G_n(-\delta\underline{1}))] \tag{3-4}$$

for all $n \geq 1$.

(D) There exists $0 < q < 1/3$, $l > 0$ such that

$$\int_{R_m} |\phi_n(\tau_n + it)/\phi_n(\tau_n)|^{l/n}\, dt = O(e^{n^q}). \tag{3-5}$$

If $f_n(x)$ denotes the probability density function of T_n/n then

$$f_n(y_n) = \frac{n^{m/2}}{(2\pi)^{m/2}\,|\nabla^2\psi_n(\tau_n)|^{1/2}}\exp(-n\gamma_n(y_n))[1+O(1/n)]. \tag{3-6}$$

Remark 3.3. The conclusion of Theorem 3.2 still holds true if we replace Condition (C) by the following Condition (C′) (see Chaganty and Sethuraman(1987)):

(C′) Given $\delta > 0$, there exists $0 < \eta < 1$ such that

$$\limsup_{n\geq 1}\ \sup_{|t|\geq\delta} |\phi_n(\tau_n+it)/\phi_n(\tau_n)|^{1/n} \leq \eta. \tag{3-7}$$

We will postpone the proof of Theorem 3.2 until the end of the following lemma.

Lemma 3.4. Let $\{T_n, n \geq 1\}$ be a sequence of nonlattice random vectors taking values in R_m. Assume that conditions (A), (B) and (C) of Theorem 3.2 are satisfied. Let $G_n(t) = [\psi_n(\tau_n) + i < t, y_n > -\psi_n(\tau_n + it)]$. Then there exists $\delta_1 < \eta$ such that for $0 < \delta < \delta_1$,

$$\inf_{|t|\geq n^\delta} Real(G_n(t)) \geq \beta_2 m\delta^2/4, \quad \text{for all} \quad n \geq 1. \tag{3-8}$$

Proof. Since ψ_n is a well defined holomorphic function in Ω^m and τ_n is in I_1^m for all $n \geq 1$, the following expansion is valid for $|t| < (a-a_1)/2$,

$$\psi_n(\tau_n+it) = \psi_n(\tau_n)+i<t,\nabla\psi_n(\tau_n)> -\frac{1}{2}t'\nabla^2\psi_n(\tau_n)t - i\sum_{|\alpha|=3} a_\alpha^{(n)}t^\alpha + R_n(\tau_n+it), \tag{3-9}$$

where $R_n(\tau_n+it) = \sum_{|\alpha|\geq 4} a_\alpha^{(n)}(it)^\alpha$. By Cauchy's Theorem 2.3 and Condition (A) we get the following bound for $a_\alpha^{(n)}$,

$$|a_\alpha^{(n)}| \leq \frac{\beta_1}{(a-a_1)^{|\alpha|}}. \tag{3-10}$$

Thus for $|t| < (a-a_1)/2$ and for all $n \geq 1$,

$$\begin{aligned} |R_n(\tau_n+it)| &\leq \sum_{|\alpha|\geq 4}\prod_{i=1}^{m}|t_i|^{\alpha_i}\frac{\beta_1}{(a-a_1)^{|\alpha|}} \\ &\leq \frac{2\beta_1|t|^4}{(a-a_1)^4}. \end{aligned} \tag{3-11}$$

It follows from the definition of $G_n(t)$, (3-9), (3-10) and (3-11) that for $n \geq 1$,

$$\left|\frac{G_n(t)}{|t|^2} - \frac{t'\nabla^2\psi_n(\tau_n)t}{2|t|^2}\right| \leq \left|\frac{\sum_{|\alpha|=3} a_\alpha^{(n)} t^\alpha}{|t|^2}\right| + \left|\frac{R_n(\tau_n + it)}{|t|^2}\right| \tag{3-12}$$
$$\leq \frac{\beta_1|t|}{(a-a_1)^3} + \frac{2\beta_1|t|^2}{(a-a_1)^4}.$$

Therefore there exists $0 < \delta_1 < \eta$ such that for $|t| < \delta_1$ we have the inequality

$$\left|\frac{G_n(t)}{|t|^2} - \frac{t'\nabla^2\psi_n(\tau_n)t}{2|t|^2}\right| < \frac{\beta_2}{4m}, \tag{3-13}$$

which implies that

$$\left|\frac{Real(G_n(t))}{|t|^2}\right| \geq \frac{|t'\nabla^2\psi_n(\tau_n)t|}{2|t|^2} - \frac{\beta_2}{4m} \tag{3-14}$$
$$\geq \frac{\beta_2}{2m} - \frac{\beta_2}{4m} = \frac{\beta_2}{4m}.$$

We have used Condition (B) and $m|t't| \geq |t|^2$ in the last inequality. Thus, if $|t| < \delta_1$ then $Real(G_n(t)) \geq \beta_2|t|^2/4m$ for all $n \geq 1$. Now for $0 < \delta < \delta_1$ it follows from Condition (C) that

$$\inf_{|t|\geq\delta} Real(G_n(t)) = min\left[Real(G_n(\delta\underline{1})), Real(G_n(-\delta\underline{1}))\right] \tag{3-15}$$
$$\geq \beta_2 m\delta^2/4, \qquad \text{for all} \quad n \geq 1.$$

We now return to the proof of our main Theorem 3.2.

Proof of Theorem 3.2: The proof parallels the proof of Theorem 3.1 of Chaganty and Sethuraman(1987). Proceeding as in Chaganty and Sethuraman(1987) we can show that Condition(D) implies that the density function of T_n/n exists and equals

$$f_n(x) = \frac{n^m}{(2\pi)^m}\int_{R_m} \phi_n(\tau_n + it)\exp(-n < \tau_n + it, x >)\,dt. \tag{3-16}$$

Substituting $x = y_n$, we get

$$f_n(y_n) = \frac{n^m}{(2\pi)^m}\int_{R_m} \phi_n(\tau_n + it)\exp(-n < \tau_n + it, y_n >)\,dt \tag{3-17}$$
$$= \frac{n^{m/2}}{(2\pi)^{m/2}|\nabla^2\psi_n(\tau_n)|^{1/2}}\exp(-n\gamma_n(y_n))\; I_n,$$

where,

$$I_n = \frac{n^{m/2}|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{R_m} \phi_n(\tau_n + it)\exp(n[\gamma_n(y_n) - < \tau_n + it, y_n >])\, dt \tag{3-18}$$

Noting that $\gamma_n(y_n) =< \tau_n, y_n > -\psi_n(\tau_n)$, we can write

$$\begin{aligned}
I_n &= \frac{n^{m/2}|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{R_m} \exp(n[\psi_n(\tau_n + it) - \psi_n(\tau_n) - i < t, y_n >])\, dt \\
&= \frac{n^{m/2}|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \left[\int_{|t|\geq n^{-\lambda}} \exp(-nG_n(t))\, dt + \int_{|t|<n^{-\lambda}} \exp(-nG_n(t))\, dt\right] \\
&= I_{n1} + I_{n2} \qquad \text{(say)},
\end{aligned} \tag{3-19}$$

where λ is chosen to be a number such that $q < 1-2\lambda < 1/3$ and q is as in Condition(D). We shall complete the proof of the Theorem 3.2 by showing first I_{n1} goes to zero exponentially fast and then showing that $I_{n2} = [1 + O(1/n)]$. By Lemma 3.4, we can find N such that for $n \geq N$, $\inf_{|t|\geq n^{-\lambda}} Real(G_n(t)) \geq \beta_2 mn^{-2\lambda}/4$. Thus for $n \geq N$,

$$\begin{aligned}
|I_{n1}| &= \frac{n^{m/2}|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|t|\geq n^{-\lambda}} \exp(-nG_n(t))\, dt \\
&\leq \frac{n^{m/2}|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \max_{|t|\geq n^{-\lambda}} [\exp(-(n-l)G_n(t))] \int_{|t|\geq n^{-\lambda}} \exp(-lG_n(t))\, dt \\
&= O(n^{m/2}e^{n^q}) \max_{|t|\geq n^{-\lambda}} [\exp(-(n-l)Real(G_n(t)))] \\
&= O\big(n^{m/2}\exp(-\beta_2 m(n-l)(n^{-2\lambda})/4 + n^q)\big)
\end{aligned} \tag{3-20}$$

which goes exponentially fast to zero, since $0 < q < 1-2\lambda$. Substituting $s/\sqrt{n}$ for t in the second term, I_{n2} of (3-19) we get

$$\begin{aligned}
I_{n2} &= \frac{|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-nG_n(s/\sqrt{n}))\, ds \\
&= \frac{|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s + Z_n(s))\, ds \\
&= \frac{|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s)\big[1 + Z_n(s) + L_n(s)\big]\, ds
\end{aligned} \tag{3-21}$$

where $Z_n(s) = \big[-nG_n(s/\sqrt{n}) + (\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s)\big]$ and $L_n(s) = \big[\exp(Z_n(s)) - 1 - Z_n(s)\big]$. Note that $|s|/\sqrt{n}$ goes to zero uniformly in s, as $n \to \infty$, for $|s| < n^{1/2-\lambda}$, therefore for

sufficiently large n the infinite series expansion (3-9) is applicable with t replaced by $s/\sqrt{n}$. Thus we can write

$$[Z_n(s)] = [-\frac{i}{\sqrt{n}} \sum_{|\alpha|=3} a_\alpha^{(n)} s^\alpha + nR_n(\tau_n + is/\sqrt{n})] \tag{3-22}$$

and therefore for sufficiently large n,

$$\begin{aligned} I_{n2} &= \frac{|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s)\, ds \\ &- \frac{i|\nabla^2\psi_n(\tau_n)|^{1/2}}{\sqrt{n}(2\pi)^{m/2}} \sum_{|\alpha|=3} a_\alpha^{(n)} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s)\, s^\alpha\, ds \\ &+ \frac{n|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s)\, R_n(\tau_n + is/\sqrt{n})\, ds \\ &+ \frac{|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s)\, L_n(s)\, ds \end{aligned} \tag{3-23}$$

It is easy to verify that the first term on the right hand side is equal to $[1 + o(1/n)]$ and the second term equal to zero. The proof is completed by showing that the last two terms are equal to $O(1/n)$. Since $|s|/\sqrt{n}$ goes to zero uniformly in s, for $|s| < n^{1/2-\lambda}$, there exists N_1 such that for $n \geq N_1$, we have from (3-11) with t replaced by $s/\sqrt{n}$,

$$|R_n(\tau_n + is/\sqrt{n})| \leq \frac{2\beta_1|s|^4}{n^2(a-a_1)^4}. \tag{3- 24}$$

Therefore for $n \geq N_1$,

$$\begin{aligned} &\left|\frac{n|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s)\, R_n(\tau_n + is/\sqrt{n})\, ds\right| \\ &\leq \frac{n|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s)\, |R_n(\tau_n + is/\sqrt{n})|\, ds \\ &\leq \frac{2\beta_1|\nabla^2\psi_n(\tau_n)|^{1/2}}{n(2\pi)^{m/2}(a-a_1)^4} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2}s'\nabla^2\psi_n(\tau_n)s)\, |s|^4\, ds \\ &= O(1/n). \end{aligned} \tag{3-25}$$

To show that the fourth term on the right hand side of (3-23) is $O(1/n)$, we first make the simple observation that if $|z| < 1/2$, for complex z, then $|\exp(z) - 1 - z| < 6|z|^2$. This inequality will be used to get an upper bound for $L_n(s)$. Combining (3-22), (3-10) and (3-24) we get

$$
\begin{aligned}
|Z_n(s)| &= |-\frac{i}{\sqrt{n}} \sum_{|\alpha|=3} a_\alpha^{(n)} s^\alpha + nR_n(\tau_n + is/\sqrt{n})| \\
&\leq \frac{|s|^3 \beta_1}{\sqrt{n}(a - a_1)^3} + \frac{2|s|^4 \beta_1}{n(a - a_1)^4}, \\
&\leq \frac{\beta_1 n^{1-3\lambda}}{(a - a_1)^3} + \frac{2\beta_1 n^{1-4\lambda}}{(a - a_1)^4},
\end{aligned}
\tag{3-26}
$$

for $|s| \leq n^{1/2-\lambda}$. Since $\lambda > 1/3$ the r.h.s. of (3-26) converges to zero as $n \to \infty$. Thus we can find N_2 such that for $n \geq N_2$, $|Z_n(s)| < 1/2$ for all $|s| \leq n^{1/2-\lambda}$ and hence

$$
\begin{aligned}
|L_n(s)| &\leq 6|Z_n(s)|^2 \\
&\leq 6\left[\frac{|s|^3 \beta_1}{\sqrt{n}(a - a_1)^3} + \frac{2|s|^4 \beta_1}{n(a - a_1)^4}\right]^2.
\end{aligned}
\tag{3-27}
$$

Therefore for $n \geq N_2$,

$$
\begin{aligned}
&\left| \frac{|\nabla^2 \psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2} s' \nabla^2 \psi_n(\tau_n) s)\, L_n(s)\, ds \right| \\
&\leq \frac{|\nabla^2 \psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2} s' \nabla^2 \psi_n(\tau_n) s)\, |L_n(s)|\, ds \\
&\leq \frac{6|\nabla^2 \psi_n(\tau_n)|^{1/2}}{n(2\pi)^{m/2}} \int_{|s|<n^{1/2-\lambda}} \exp(-\frac{1}{2} s' \nabla^2 \psi_n(\tau_n) s) \left[\frac{|s|^3 \beta_1}{(a - a_1)^3} + \frac{2|s|^4 \beta_1}{\sqrt{n}(a - a_1)^4}\right]^2 ds \\
&= O(1/n).
\end{aligned}
$$

The proof of Theorem 3.2 is now complete.

Remark 3.5. We now compare the conditions imposed in Theorem 3.1 and Theorem 3.2. We have already remarked that Theorem 3.2 is valid for sequences $\{y_n\}$ which may not converge to zero. Condition (A) is similar to Condition(b). Condition (B) and Condition (a) are the same if $y_n \to 0$. Conditions (C) and (D) imply Condition (c) when $y_n \to 0$. Thus Theorem 3.2 not only generalizes Theorem 3.1 but also weakens Condition (c).

Remark 3.6. We only require the weaker condition that the eigenvalues of $\nabla^2\psi_n(\tau_n)$ are bounded below by β_2 for all $n \geq 1$, in the proof of Theorem 3.2. The stronger Condition (B) of Theorem 3.2 allows us to obtain further refinements of the expression (3-6) as stated in the next corollary.

Corollary 3.7. Let $\{T_n, n \geq 1\}$ be a sequence of nonlattice random vectors satisfying conditions (A), (B), (C) and (D) of Theorem 3.2 for some sequence $\{y_n\}$ of random vectors in R_m. Suppose that y_n converges to y as $n \to \infty$ and $n^\delta \parallel y_n - y \parallel \geq 1$, for some $0 < \delta < 1$ and for all $n \geq 1$. Let $E(T_n/n) = y$ and $Cov(T_n/\sqrt{n}) = \Sigma_n$ be a positive definite matrix. Then

$$f_n(y_n) = \frac{n^{m/2}}{(2\pi)^{m/2}|\Sigma_n|^{1/2}} \exp(-n\gamma_n(y_n))[1 + O(\parallel y_n - y \parallel)] \tag{3-28}$$

The proof of Corollary 3.7 is identical to the proof of Corollary 3.8 of Chaganty and Sethuraman(1987) and hence is omitted.

4. Large deviation local limit theorems for lattice random vectors

The study of large deviation local limit theorems for i.i.d. lattice random variables was initiated by Richter(1957). Since then numerous authors including Moskvin(1972), McDonald(1979) have extended these results for sums of independent, not necessarily identically distributed lattice random variables. Recently Chaganty and Sethuraman(1985) obtained generalizations to arbitrary sequences of lattice random variables and applied these results for statistics appearing in nonparametric inference. In a subsequent paper, Richter(1958) proved analogous theorems in the multi-dimensional case. The purpose of this section is to obtain local limit theorems for arbitrary lattice valued random vectors in the area of large deviations analogous to the results of Section 3. We begin with some notation.

Let $(e_1, \ldots, e_m)$ be a basis for R_m. Let L_n be the lattice $\{\xi : \xi = h_n(n_1 e_1 + \cdots + n_m e_m),\ n_i's \text{ are integers }\}$, where $\{h_n\ n \geq 1\}$ is a sequence of real numbers. Let $\{T_n,\ n \geq 1\}$ be an arbitrary sequence of nondegenerate random vectors defined on the lattice L_n. Let $\{y_n\}$ be a sequence of vectors such that ny_n is in L_n, for all $n \geq 1$ and let ϕ_n, ψ_n be as defined as in Section 3. Assume that $\phi_n(z)$ is analytic in the region Ω^m, where $\Omega = \{x + iy : x \in (-a, a), y \in R\}$. For the lattice random vectors T_n we have the following theorem, which is analogous to Theorem 3.2 of Section 3.

Theorem 4.1. Let T_n be a lattice valued random vector taking values in L_n, for $n \geq 1$. Assume that Conditions (A), (B) of Theorem 3.2 and Condition (C$''$) stated below hold for a sequence $\{y_n, n \geq 1\}$ of vectors, where $n\, y_n \in L_n$.

(C$''$) There exists $\eta > 0$ such that for any δ, $0 < \delta < \eta$,

$$\text{(4-1)} \qquad \inf_{\delta \leq |t_j| \leq \pi/|h_n|} Real(G_n(t)) = min[Real(G_n(\delta \underline{1})),\ Real(G_n(-\delta \underline{1}))]$$

for all $n \geq 1$, where $G_n(t) = \psi_n(\tau_n) + i < t, \nabla\psi_n(\tau_n) > -\psi_n(\tau_n + it)$ and $\nabla\psi_n(\tau_n) = y_n$. Further assume that the span h_n of the lattice L_n is equal to $O(n^{-p})$ for $p > 0$. Let $\gamma_n(u) = \sup_{s \in I^m}[< u, s > -\psi_n(s)]$. Then

$$\text{(4-2)} \qquad \frac{n^{m/2}}{|h_n|^m} Pr(T_n = ny_n) = \frac{1}{(2\pi)^{m/2} |\nabla^2 \psi_n(\tau_n)|^{1/2}} \exp(-n\gamma_n(y_n)) \left[1 + O(1/n)\right].$$

Proof. Let τ_n be such that $\nabla\psi_n(\tau_n) = y_n$ for $n \geq 1$. By definition

(4-3)
$$\phi_n(\tau_n + it) = \sum_{\xi \in L_n} \exp(< \tau_n + it, \xi >)\, Pr(T_n = \xi),$$

for $(\tau_n + it) \in \Omega^m$. Multiplying both sides by $\exp(- < \tau_n + it, ny_n >)$ and integrating over the region $B_n = \{(t_1, \ldots, t_n) : |t_j| < \pi/|h_n|, \text{for all } j\}$, we get

(4-4)
$$\begin{aligned} Pr(T_n = ny_n) &= \frac{|h_n|^m}{2\pi^m} \int_{B_n} \exp(-n < \tau_n + it, y_n >)\, \phi_n(\tau_n + it)\, dt \\ &= \frac{|h_n|^m}{n^{m/2}(2\pi)^{m/2}|\nabla^2\psi_n(\tau_n)|^{1/2}} \exp(-n\gamma_n(y_n))\, I_n, \end{aligned}$$

where

(4-5)
$$\begin{aligned} I_n &= \frac{n^{m/2}|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{B_n} \exp(n\gamma_n(y_n) - n < \tau_n + it, y_n >)\, \phi_n(\tau_n + it)\, dt \\ &= \frac{n^{m/2}|\nabla^2\psi_n(\tau_n)|^{1/2}}{(2\pi)^{m/2}} \int_{B_n} \exp(-nG_n(t))\, dt. \end{aligned}$$

Imitating the proof of Theorem 3.2 and noting that

(4-6)
$$\int_{B_n} \left| \frac{\phi(\tau_n + it)}{\phi_n(\tau_n)} \right|^{1/n} dt \le \frac{(2\pi)^m}{|h_n|^m} = O(n^p),$$

we can show that $I_n = 1 + O(1/n)$. The proof of Theorem 4.1 is completed substituting this in (4-4).

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